Economics of Prison: Modeling the Dynamics of Recidivism

Alejandra Lopez¹, Naomi Moreira², Ariana Rivera³, Bechir Amdouni¹, Baltazar Espinoza¹ and Christopher M Kribs⁴

¹Arizona State University
²St. Joseph’s College
³Yachay Tech University
⁴University of Texas at Arlington

Abstract

In Arizona each prisoner costs the state an average of $25,397 per year, the approximate cost of attending Arizona State University [21]. Based on the current population of inmates this adds up to over 1 billion dollars annually. This figure is 5 times more than is spent on public assistance and about 70 percent of what is spent on transportation in the entire state. In addition to that, 40 percent of those who leave the prison return which further increases the costs on the state. In an effort to decrease costs the government of Arizona hopes to implement programs into their prison system in order to lower the recidivism rate and decrease costs. Multiple studies have shown recidivism is reduced when education and transition programs are incorporated. Currently, Arizona funds most of the GED program, excluding testing, while an inmate is incarcerated. Unfortunately, this education does not continue after the inmate is released. Meanwhile, other states have successfully incorporated education in order to reduce recidivism. In an effort to analyze recidivism in Arizona we have developed and analyzed a data-driven mathematical model that captures the dynamics of prisoners while in and out of prison based on their education status. This model, a system of differential equations, helped to estimate the cost associated with different educational programs in and outside of prison to the cost of recidivism. As a result we were able to study the economic impact of implementing these transition programs which we proved to be cost efficient. We found that the transition programs would eventually pay for themselves as a higher proportion of inmates enroll in the program. We were also able to show that it was possible to completely eliminate recidivism as the length of the program increased and enough inmates enrolled in the transition program after being released.

1 Introduction

In 1984, the United States entered the “tough on crime” era which resulted in mandatory minimum sentences for drug offenses [17]. As a result of this measure, incarceration rates increased rapidly over a short period of time. Presently, the United States has the highest incarceration rate per year in the world with over 2 million people imprisoned [23]. That is roughly 6 times higher than China’s incarceration rate per year, a country with a population 4 times larger than the population of the United States. [3] Incarceration and high recidivism rates come with high economic and social costs. Consequently, various states have invested a great amount of effort into minimizing the economic impact on their budget. We consider how effective educational correction programs are, specifically the General Education Diploma (GED) inside of the state prisons and the reentry programs outside that parolees have access to in Arizona.

Over the last thirty years, the inmate population in Arizona has increased from over ten thousand inmates to more than forty two thousand inmates, an increase of 293%. To accommodate the increased population of inmates, the corrections’ budget also increased dramatically over the same time period going from $211.5 million to $1.1 billion, an increase of 419% [18]. This hefty price tag is of major concern to the Arizona Department of Corrections which stated that the level of historical growth is unsustainable [15]. Many states are targeting transition and recidivism reduction initiatives as a more effective criminal justice investment than continuing to expand prison populations and construct new prison beds [18]. While it is clearly in the best interest of the Arizona Department of Corrections to reduce the rate of recidivism it is also in the best interest of the community;this would mean that the
money saved could eventually be reinvested into other areas of the community that are underfunded such as education and infrastructure.

Arizona Department of Corrections published their 2018-2022 plan which includes a strategy to reduce recidivism through reentry preparation and support [19]. The current recidivism rate for Arizona inmates is 39.1% per year [20]. The strategies to reduce it include leading the state-level breakthrough project on recidivism reduction, improving inmate programs, increasing emphasis on inmate completions of degree, stabilizing inmates’ mental health needs prior to release, increasing community engagement, and improving use of offender interventions and sanctions [20]. As an evaluation, the RAND (Research and Development) corporation conducted a study that evaluated the effectiveness of correctional education. Among the key findings, they found that correctional education improves inmates’ chances of not returning to prison. Inmates who participate in correctional education programs had 43% lower odds of recidivating than those who did not, this translates to a reduction in the risk of recidivating of 13 percentage points. In other words, providing correctional education can be cost-effective when it comes to reducing recidivism [3].

In this study we focus on two types of educational programs; the first is the GED program offered in all of the state prisons while the second is the Reentry Court Program. The GED is offered to inmates who have completed the Functional Literacy program, an equivalence to having an 8th grade education [12]. The Reentry Court Program helps a parolee find a job and even offers temporary housing. In this study, we propose a mathematical approach to examine under what conditions the available program to inmates and the one available to parolees are the most cost effective and reduce recidivism in Arizona.

The goal of the mathematical study is to examine the qualitative differences between the GED and non-GED holders in prison so we can understand how to best invest monetary funds. Mechanically, inmates go through the same process but in order to really establish the effectiveness of education, we look at the qualitative journey which is different for each group. With an analogy to epidemiology where disease transmission is modeled, we use a compartmental and deterministic model to analyze mechanisms inside a given prison and inmates’ influences on transmission of education as an infectious social disease where “education” is the disease. In this study, we only consider the Arizona population that is incarcerated in state prisons and exclude the private prisons because we don’t have enough access to enough data. For simplicity we analyze the inmates with or without GED, with or without the transition program, and with or without recidivating.

A previous study in 2012 by Alvarez et al. [1] used a mathematical model to analyze how recidivism rates were impacted based on which reform programs the inmate attended. They concluded that recidivism rates lowered if inmates had the opportunity to go through both the inside and the outside programs. The study only analyzed data from California therefore has some geographical limitations. However, it did not employ a cost analysis of the economic impact that these programs had on the corrections budget.

In 2015, Purtolas et al modeled the direct impact of incentivized educational programs on the recidivism rates in Louisiana [6]. The study focused on finding out how much incentive the state has to provide in order to reduce recidivism. The claim is that in order for a prison’s optimal profit strategy to reduce recidivism then an incentive has to be offered. The study found that more effective reform programs have a more cost effective strategy in reducing recidivism. However, the study found that the way the current prison system operates the prison is highly motivated to reduce the effectiveness of the reform programs.

Our model builds on the studies previously conducted and will use many of the same concepts in order to model recidivism in Arizona and conduct a cost analysis. Furthermore, our study places great emphasis on the available transition program that inmates have access once they are released.

We will consider programs that occur outside and inside prison. The program we focus on is the GED program that takes place inside prison where inmates can take classes in order to earn their degree and the transition program outside of prison that helps paroled inmates reintegrate into society. The GED program prepares inmates for life outside of prison and they are also a prerequisite for them to participate in the work programs offered inside of prison [12]. The GED program is mostly funded through the state except for the testing that comes at the inmates’ expense. Both programs are completely optional to the parolee and they do not have to complete them. Studies have shown that inmates who complete both programs are less likely to return to prison [14]. The analysis of the model shows effectiveness of the education program inside and the transition program outside. We hypothesize that in order to reduce the levels of recidivism, the inmates of the Arizona state prisons would need to pass through the prison programs offered by the state during their sentence and after they are released. The following sections developed the ideas discussed here. In section 2, we discuss the methods used to develop our
2 Methods

2.1 Model

The proposed model employs a system of ordinary differential equations, tracking the number of inmates within one prison that enter without GED and the number of prisoners with GED or higher educational degrees in a single prison. We then look at the likelihood of the inmate to recidivate once he is released. Our main objective is to find under what conditions are the available education programs, such as the GED and Reentry centers in Arizona, the most cost effective in reducing recidivism. We hypothesize that in order to reduce the levels of recidivism, the inmates of the Arizona state prisons would need to pass through the prison programs offered during and after their prison sentence.

2.2 Prison and Education Dynamics

In our model, the inside prison dynamics are given by the first-time offenders class which is divided into two parts: $I_1$ represents the first-time inmates that come in to prison without a GED and $I_2$ are those that are in prison for the first-time with a GED or higher education levels. The $E$ compartments represent the inmates that have come back to prison after having already been freed once. These compartments are also divided into two parts: $E_1$ repeat offenders without GED, and $E_2$ repeat offenders with GED or more. These compartments allow us to keep the first time offenders separate from the ones that recidivated to help us calculate the cost. The transition rates between compartments on the inside are defined by contact rates between the uneducated and educated, assuming that the educated influence the uneducated to complete their GED. In the $T$ compartments which are outside prison, $T_1$ represents the released inmates that were not involved in the GED program, but are involved in the reentry program. Class $T_2$ are the inmates that either have a GED or received one while inside and go to the reentry program. The $O$ compartments are divided into: $O_1$, the released inmates not currently in a transitional program (and still at risk); and $O_2$, the released inmates that completed the GED program and are not currently in a transitional program (and still at risk). Even though the $T$ and $O$ compartments appear to be similar, the proportion of released inmates that exit from the $T$ compartment are our ideal released inmate that leave the reentry program and never commit another crime. The $O$ compartments are designed to catch the proportion of offenders that remain at risk of committing another crime. As a result, the exit rate from the $T$ compartments is larger than the $O$ compartments because we assume that the reentry program has a higher rate of effectiveness.

The proposed model assumes a constant recruitment rate ($\Lambda_i$, $i=1,2$) for each of the incoming populations of inmates ($I_i$), taking into account the incoming inmates with no GED, and the incoming inmates with a level of education of GED or more. It is assumed that a proportion ($q_i$) of inmates who leave prison after $\frac{1}{\alpha_i}$ days will go through the transition program ($T_i$), while $(1-q_i)$ remain at risk of recidivism ($O_i$). Among those released inmates, a fraction $\left(\frac{\mu_1,2}{\gamma_1+\mu_1,2}\right)$ will be rehabilitated due to the effectiveness of the program, while the rest $\left(\frac{\gamma_1}{\gamma_1+\mu_1,2}\right)$ will remain at risk of recidivism. While the inmates are at risk of recidivism, they can go back to prison at a per capita rate $p_i$ or they can rehabilitate at per capita rate $\mu_{3,4}$.

Our model looks at how peer influence between both of these groups (the group with no GED and the group with GED or more) affect the population density in the different compartments. The incoming population of inmates with no GED is represented by $\Lambda_1$ and the incoming population of inmates with GED or more is represented by $\Lambda_2$. Even though both of the prisoner classes can attend a transition program when released, released inmates with high education levels have less chance of recidivism \cite{5}. In this case, we are assuming that the infection represents the positive influence that the population with GED or more has over the population with no GED.

The classes and parameters of the model are explained in further detail in Tables 1, 2.
Table 1: Definition of variables in the model

<table>
<thead>
<tr>
<th>Class</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1$</td>
<td>Inmates with no GED</td>
</tr>
<tr>
<td>$I_2$</td>
<td>Inmates with GED or more</td>
</tr>
<tr>
<td>$E_1$</td>
<td>Returning offenders without GED</td>
</tr>
<tr>
<td>$E_2$</td>
<td>Returning offenders with GED or more</td>
</tr>
<tr>
<td>$T_1$</td>
<td>Released inmates without GED attending the transition program</td>
</tr>
<tr>
<td>$T_2$</td>
<td>Released inmates with GED attending the transition program</td>
</tr>
<tr>
<td>$O_1$</td>
<td>Inmates without GED who are still at risk to recidivate</td>
</tr>
<tr>
<td>$O_2$</td>
<td>Inmates with GED who are still at risk to recidivate</td>
</tr>
</tbody>
</table>

Table 2: Description of the parameters used in the model

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_1$</td>
<td>Incoming population without GED</td>
<td>People per time</td>
</tr>
<tr>
<td>$\Lambda_2$</td>
<td>Incoming population with GED or more</td>
<td>People per time</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>Rate of influence between $I_2$ and $I_1$</td>
<td>Per person per time</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>Rate of influence between $E_2$ and $E_1$</td>
<td>Per person per time</td>
</tr>
<tr>
<td>$p_1$</td>
<td>Rate of people leaving $O_1$</td>
<td>1/time</td>
</tr>
<tr>
<td>$p_2$</td>
<td>Rate of people leaving $O_2$</td>
<td>1/time</td>
</tr>
<tr>
<td>$q_1$</td>
<td>Proportion of released inmates without GED who go to the transition program</td>
<td>Dimensionless</td>
</tr>
<tr>
<td>$q_2$</td>
<td>Proportion of released inmates with GED or more who go to the transition program</td>
<td>Dimensionless</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>Per capita rehabilitation rate from $T_1$</td>
<td>1/time</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>Per capita rehabilitation rate from $T_2$</td>
<td>1/time</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>Per capita rehabilitation rate from $O_1$</td>
<td>1/time</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>Per capita rehabilitation rate from $O_2$</td>
<td>1/time</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>Rate of people leaving $T_1$</td>
<td>1/time</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>Rate of people leaving $T_2$</td>
<td>1/time</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>Rate of people leaving $I_1$</td>
<td>1/time</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>Rate of people leaving $I_2$</td>
<td>1/time</td>
</tr>
</tbody>
</table>

The system of ordinary differential equations (1) captures the aforementioned prison education dynamics. It is also represented in Figure 1.

\[
\begin{align*}
\frac{dI_1}{dt} &= \Lambda_1 - \beta_1 I_1 (I_2 + E_2) - \alpha_1 I_1 \\
\frac{dI_2}{dt} &= \Lambda_2 + \beta_1 I_1 (I_2 + E_2) - \alpha_2 I_2 \\
\frac{dE_1}{dt} &= p_1 O_1 - \beta_2 E_1 (I_2 + E_2) - \alpha_1 E_1 \\
\frac{dE_2}{dt} &= p_2 O_2 + \beta_2 E_1 (I_2 + E_2) - \alpha_2 E_2 \\
\frac{dT_1}{dt} &= (I_1 + E_1) (q_1 \alpha_1) - T_1 (\mu_1 + \gamma_1) \\
\frac{dT_2}{dt} &= (I_2 + E_2) (q_2 \alpha_2) - T_2 (\mu_2 + \gamma_2) \\
\frac{dO_1}{dt} &= \gamma_1 T_1 + (1 - q_1) (\alpha_1 I_1 + \alpha_1 E_1) - O_1 (p_1 + \mu_3) \\
\frac{dO_2}{dt} &= \gamma_2 T_2 + (1 - q_2) (\alpha_2 I_2 + \alpha_2 E_2) - O_2 (p_2 + \mu_4)
\end{align*}
\]
3 Parameters

Our parameters as defined in Table 2 can mostly be calculated from the data that have been gathered during our investigation. When calculating the parameters, we only focused on one prison therefore the total population comes from the mean of the total population of the Florence State Prison which is 3,955 [11]. Our rates of movement from one compartment to another are rescaled to fit this population. We also focused on the male population older than 18 years of age. While some values were calculated in a pretty straightforward manner others required estimation as shown below.

3.1 Parameter Calculations

On calculating $\alpha_1$ and $\alpha_2$, we supposed that the exit rates are the same regardless of GED. As stated on the Arizona Department of Corrections website, the average prison duration term is 24 months [11]. Therefore:

$$\alpha_1 = \alpha_2 = \frac{1}{\text{Average Length of Stay}} = \frac{1}{24} = 0.04 \text{ /month}$$

The direct data for Arizona results were not found for the number of inmates coming in with a GED. Therefore we used the 2016 data from Florida’s Department of Corrections statistics on TABE, the test of adult basic education, to calculate $\Lambda_i$, the incoming population with a GED [15]. We decided to use data from Florida since they have data available about their prisoner’s education level. We find that 44% of inmates that are being admitted, have no priors, and have GED Literacy [16]. We then calculated the rate at which male inmates with no priors are being admitted in Arizona. This will be an estimated calculation since we used data from Florida, since.
Males without priors admitted per year = \frac{x}{1 \text{ month}},
\begin{align*}
10,236 \text{ people} &\div 12 \text{ months} = \frac{x}{1 \text{ month}}, \\
x & = 853 \text{ admissions per month},
\end{align*}
\begin{align*}
853 \text{ people} \times .44 & = 478 \text{ people} \div 1 \text{ month}, \\
\Lambda_2 & = 478 \text{ people per month},
\end{align*}
\begin{align*}
\Lambda_1 & = 375 \text{ people per month}.
\end{align*}

\(p_1\) represents the rate of recidivism of those without GED. We calculate this rate from a recidivism study done in Arizona in 2005 \cite{10} which shows that 42.4\% of parolees returned to the Arizona Department of Corrections custody for any reason over a 3 year period. Using this we calculate the rate of recidivism in Arizona. We will rescale this percentage to reflect our month scale.

\[
p_1 = \frac{\text{Recidivism Rate}}{3 \text{ years}},
\]
\[
p_1 = \frac{.424}{36 \text{ months}},
\]
\[
p_1 = 0.0118 \text{ per month}.
\]

In order to calculate \(p_2\) (the rate of recidivism of those with GED or more) we consider the same study used to derive \(p_1\). The study found that those who participate in inmate programs like academic education reduce the recidivism rates by an average of 25\% \cite{10}. As a result, we used the same calculations but using a recidivism rate of 31.8\% which is 25\% lower.

\[
\text{New percentage} = \text{recidivism rate} - (\text{recidivism rate} \times \text{percentage lowered}) = \text{new percentage}
\]
\[
(0.424 - (0.424 \times 0.25)) \times 100 = 31.8\%
\]

Now running through the same calculations as \(p_1\) using our new percentage we find that
\[
p_2 = \frac{0.318}{36 \text{ months}},
\]
\[
p_2 = 0.009 \text{ per month}.
\]

### 3.2 Parameter Estimation

In this section we estimated the parameters \(\beta_1\), \(\mu_1\), \(\gamma_1\), and \(q_i\) since the meaning of these parameters involved various methods that allow them to fit our assumptions.

We will begin with the estimation of \(\mu_1\) and \(\gamma_1\). The basic structure of the model creates two unique exit rates that represent the same thing; \(\mu_1\) is the exit rate from the transition program that assumes that the parolee will cease to recidivate, and \(\gamma_1\) is incorporating the latency period where the parolee can still recidivate and become an inmate again. As a result, the calculated rate at which a parolee leaves the transition program will be split amongst the unknown probability of the program success. We begin by calculating the rate at which people leave the transition program which we will call \(\eta\).

As mentioned before the main difference between our study and previously conducted studies is that we analyzed the impact of having a transition program once the inmates are released. Currently, Arizona has the Maricopa Reentry Center and the Pima Reentry center. According to a table presented in February 2018 to the Appropriations Committee by the Arizona Department of Correction we calculate \(\eta\), the rate at which people leave the transition program \cite{13}. The transition program is currently, at a maximum, 90 days long.

\[
\eta_1 = \frac{1}{3 \text{ months}},
\]
\[
\eta_1 = 0.3 \text{ per month},
\]
Now in order to incorporate this exit rate properly, we introduced the variable $a$ which will be the probability of the program’s success, a number for which we have no data. Using this $a$ we can define $\mu_1$ and $\gamma_1$.

$$
\mu_1 = a \times \eta,
$$

$$
\gamma_1 = (1 - a) \times \eta.
$$

We multiply $a$ by $\mu_1$ with the assumption that the program will have a slightly greater success. In order to begin the process, we will arbitrarily let $a = \frac{1}{e}$ which is about a 37% success rate. Therefore,

$$
\mu_1 = 0.12 \text{ per month},
$$

$$
\gamma_1 = 0.21 \text{ per month}.
$$

In this case, we will suppose that $\gamma_1 = \gamma_2$. Therefore $\gamma_2 = 0.21 \text{ per month}$.

The rest of the exit rates $\mu_2$, $\mu_3$, and $\mu_4$ are defined with respect to $\mu_1$.

**Assumption 1:** GED parolee’s have a double chance of reforming completely. This yields the following result:

$$
\mu_2 = 2\mu_1 = 0.24 \text{ per month}
$$

**Assumption 2:** Non-GED parolees have half the chance of reforming without the transition program.

$$
\mu_3 = \frac{1}{2}\mu_1 = 0.06 \text{ per month}
$$

**Assumption 3:** GED parolees that do not go through the GED program have the same chance as non-GED parolees that do go through the transition program.

$$
\mu_4 = \mu_1 = 0.12 \text{ per month}
$$

In order to calculate $\beta_1$, we gathered the data of total monthly enrollments for the GED program from the Arizona Department of Corrections reports [11]. We compiled the data into a list of data points and created a graph of the overall enrollment for a total of three years which is typically the length of time used to measure recidivism.

In our model we assume that the social influence, denoted as $\beta_1$, is the only way other inmates are motivated to sign up for the GED classes. Therefore, we expect $\beta_1$ to reflect our collected data. In order to achieve this, we used a Monte Carlo Fitting on our data. We consider a simple S-I-R model to get an approximate value of $\beta_1$ that will best estimate the data that we have currently. We also supposed that the length of time that the infected individual has to influence the susceptible one is the average release rate [11]. After running our simulation of the SIR model under mass incidence conditions, we found that $R_e = 1.175$ (see fig. 2). In our simplified model $R_e = \frac{N \beta}{\gamma}$. In our model $\gamma = \text{average rate of release for inmate} = 0.04 \text{ per month}$, $N = \text{Total Population of prison}$. Therefore $\beta = \frac{\gamma R_e}{N} = 0.000012 \text{ per month}$.

In our model, the $\beta$ effect is split between the $I$ classes and the $E$ classes but the $\beta$ that was calculated using the simplified version was not. As a result, we make the assumption that the recidivists will have less motivational powers on other inmates which can be expressed as $\beta_1 > \beta_2$. Since we have no available data for social influence, the division of our calculated $\beta$ will be completely arbitrary.
Finally, we estimated the values of $q_1$ and $q_2$, the proportion of released GED and non-GED holders enrolling in a reentry program. In this case we look at the Maricopa Transition Center which has a maximum capacity of 100 beds and maximum length of stay of 90 days. We begin by calculating the number of beds available per month.

$$\text{Beds Available per month} = \frac{100 \text{ beds}}{3 \text{ months}} = \frac{33 \text{ beds}}{1 \text{ month}}$$

Then we looked at the average number of people released per month and multiplied it by the number of beds available per month to get our maximum proportion of $q$.

$$q_m = \frac{33 \text{ beds}}{1 \text{ month}} \times \frac{1}{126 \text{ people}} = 0.3$$

In order to calculate $q_1$ and $q_2$ we define a new equation that takes into account the flow rates of inmates. Therefore if there are very few people from $I_1$ and $E_1$ released then more inmates from $I_2$ and $E_2$ will be recruited.

$$q_1 \times \alpha_1 + q_2 \times \alpha_2 = q_m \times (\alpha_1 + \alpha_2)$$

We make the assumption that inmates without a GED will need more help and therefore we pick the numerical value of $q_2$ and let that define $q_1$ which will be larger than $q_m$.

$$q_2 = 0.3$$

$$q_1 = 0.4$$
4 Analysis

4.1 $R_e$ Analysis

After conducting model analysis under our starting assumptions, we found that our model has an education-free equilibrium (EFE) (see appendix A). In order to measure the strength of peer influence on the education rate, we consider the special case where $\Lambda_2 = 0$, i.e., no one comes in educated. Only then we find our EFE (in which $I_2^e = E_2^e = T_2^e = O_2^e = 0$) which yields the $R_e$, the basic reproductive number. Then we find the EFE values, including $I_1^*$ and $E_1^*$, which appear in the expression $R_e$ for $R_e$.

The education reproductive number, $R_e$, is computed using the next generation operator method [22] with the following vectors:

$$F = \begin{pmatrix} \beta_1 I_1 (E_2 + I_2) + \Lambda_2 \\ \beta_2 E_1 (E_2 + I_2) \\ 0 \\ 0 \end{pmatrix},$$

$$V = \begin{pmatrix} \alpha_2 I_2 (- (1 - q_2)) - q_2 \alpha_2 I_2 \\ -\alpha_2 E_2 (1 - q_2) - \alpha_2 E_2 q_2 + O_2 p_2 \\ \alpha_2 E_2 q_2 + \alpha_2 I_2 q_2 + (- \gamma_2) T_2 - \mu_2 T_2 \\ \alpha_2 E_2 (1 - q_2) + \alpha_2 I_2 (1 - q_2) - \mu_4 O_2 - O_2 p_2 + \gamma_2 T_2 \end{pmatrix}.$$

Here, the $F$ vector is composed of the rates at which new inmates are admitted into our initial compartments, and the $V$ vector has the transfers of individuals between compartments. We are then able to compute the respective matrices $F$ and $V$:

$$F = \begin{pmatrix} \beta_1 I_1 & \beta_1 I_1 & 0 & 0 \\ \beta_2 E_1 & \beta_2 E_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & -p_2 \\ -\alpha_2 q_2 & -\alpha_2 q_2 & \gamma_2 + \mu_2 & 0 \\ \alpha_2 (q_2 - 1) & \alpha_2 (q_2 - 1) & -\gamma_2 & \mu_4 + p_2 \end{pmatrix}.$$

Following the last steps of the next generation matrix [22] (all the steps are shown in Appendix A.1), we found a matrix with 2 rows that are multiples of each other, and the 2 others are 0 rows:

$$\begin{pmatrix} I_1 \beta_1 + \frac{I_2 p_2 (\gamma_2 - q_2 \mu_2 + \mu_4)}{\alpha_2} & \frac{I_2 \beta_1 (\gamma_2 + \mu_2) (\mu_2 + \mu_4)}{\alpha_2} \\ \frac{I_2 p_2 (\gamma_2 - q_2 \mu_2 + \mu_4)}{\alpha_2} & \frac{I_2 \beta_1 (\gamma_2 + \mu_2) (\mu_2 + \mu_4)}{\alpha_2} \end{pmatrix} = \begin{pmatrix} I_1 \beta_1 \gamma_2 & \frac{I_1 \beta_1 \gamma_2}{\alpha_2} \\ \frac{I_1 \beta_1 \gamma_2}{\alpha_2} & -\frac{I_1 \beta_1 \gamma_2}{\alpha_2} \end{pmatrix}.$$

The education reproduction number is therefore given by the largest eigenvalue of the matrix above. After some simplifications, we find:

$$R_e = \frac{I_1 \beta_1 + E_1^* \beta_2}{\alpha_2 (1 - \frac{p_2}{p_2 + \mu_4} (1 - q_2 * \frac{\mu_2}{\gamma_2 + \mu_4}))} \tag{2}$$

$$R_e = \frac{I_1 \beta_1 + E_1^* \beta_2}{\alpha_2 \frac{1}{1 - \frac{p_2}{p_2 + \mu_4} (1 - q_2 * \frac{\mu_2}{\gamma_2 + \mu_4})}}.$$

All the calculations are shown in Appendix A.1.

In order to explain what we got, we define $r_2 = \frac{p_2}{p_2 + \mu_4} (1 - q_2 * \frac{\mu_2}{\gamma_2 + \mu_4})$ which can be interpreted as the proportion of released inmates who recidivate. The term $\frac{p_2}{p_2 + \mu_4}$ is the proportion of people from $O_2$ who go to $E_2$, multiplying by $(1 - q_2 * \frac{\mu_2}{\gamma_2 + \mu_4})$ which is the proportion of people who go to $O_2$. This path can become a loop, and furthermore if we take into account this whole terms that make up the denominator, we will realize that it behaves as a geometric series. The behavior of the terms depend on the ratio $r_2$, which can be understood as the proportion of released inmates who recidivate. Which turns out to give
us all the different paths from $E_2$ that an inmate can take in order to go back to prison, then:

$$R_e = \frac{I_1^* \beta_1 + E_1^* \beta_2}{\alpha_2} \left( \frac{1}{1 - r_2} \right)$$

(3)

At the end, the $R_e$ has the term that includes $r_2$, which represents the recidivate proportion being multiplying by the influence of inmates with GED to the other who don’t have it, divided by the frequency or rate that this happens. In addition, to analyze the stability conditions for $R_e$, we used the criteria for $R_e$ [7]. We need to keep our reproductive number $R_e > 1$ in order to keep spreading the education transmission, using the non-simplified form of our $R_e$ because is more simple. Also, it can be found explicitly in Appendix A.1:

$$R_e = \frac{(\beta_1 I_1^* + \beta_2 E_2^*) (\gamma_2 + \mu_2) (p_2 + \mu_4)}{\alpha_2 (p_2 q_2 \mu_2 + \mu_4 (\gamma_2 + \mu_2))}$$

The basic reproductive number $R_e$ of our discrete model is defined and the dynamical behavior of the model is studied. It is proved that the education free equilibrium, which is only possible if $\Lambda_2 = 0$ and is globally asymptotically unstable if $R_e < 1$, and the persistence of the model is obtained when $R_e > 1$.

It is necessary to focus in the global stability of the endemic equilibrium. Sufficient conditions for the global stability of the endemic equilibrium are established by using the comparison principle. Numerical simulations are done to show our theoretical results and to demonstrate the complicated dynamics of the model. $R_e$ is the average number of secondary infections produced by one infected individual during the entire course of infection in a completely susceptible population, in our case the infection is the education. $R_e$ often serves as a threshold parameter that predicts whether an infection dies out or keeps persistence in a population [9]. Therefore, for our model the persistence of education influence is determined by the stability of the disease free equilibrium and the existence of endemic equilibrium of model. $R_e$ is always positive if:

$$(\beta_1 I_1^* + \beta_2 E_2^*) < \frac{\alpha_2 (p_2 q_2 \mu_2 + (\gamma_2 + \mu_2) \mu_4)}{\gamma_2 + \mu_2) (p_2 + \mu_4)}$$

By forthright and mindful calculations shown in Appendix A.2 we know that model has a unique endemic equilibrium when $\Lambda_2 > 0$ and $R_e > 1$.

4.2 Cost function

The total cost incurred in implementing control measures by the education programs in the Arizona Department of corrections is modeled by the function

$$C = c_1 (E_1^* + E_2^*) + c_2 (T_1^* + T_2^*) + c_3 (E_2^* + I_2^*)$$

(4)

Where $c_1$ denotes the average cost incurred by the state of Arizona to maintain an incarcerated person per day, $c_2$ represents the cost incurred by the state of Arizona to maintain a former prisoner in a non-residential transition program and $c_3$ is the amount that is spent by the state of Arizona in GED classes per prisoner daily [8]. According to the Arizona Department of Corrections Per Capita Cost Report of FY 2017, it costs the state $68.55 per day to maintain an inmate. While inside the education classes available for inmates cost the state an average of $2.25 per day. The cost to maintain per inmate in a non-residential transition center is $10.71 per day [4].

Since the parameters are measured in months, we need to convert this values of per person per day into per person per month:

$$c_1 = \frac{\text{cost/day} \times 365}{12} = \frac{68.55 \times 365}{12} = 2085.06 \text{ per person per month}$$

$$c_2 = \frac{\text{cost/day} \times 365}{12} = \frac{10.71 \times 365}{12} = 325.76 \text{ per person per month}$$

$$c_3 = \frac{\text{cost/day} \times 365}{12} = \frac{2.25 \times 365}{12} = 68.44 \text{ per person per month}$$
To estimate the numbers for the values of \( E^*_1, T^*_1, E^*_2, T^*_2, \) and \( I_2 \), we would need to use the endemic equilibrium equations and their numerical values, this section can be found on Appendix A.2. When we evaluate for each parameter we find that the value of \( E^*_1 \) is \( \approx 89 \), \( E^*_2 \approx 704 \), \( T^*_1 \approx 98 \), \( T^*_2 \approx 372 \), and \( I_2 \approx 19,237 \). Replacing these values in the original equation we get:

\[
C = 2085.06(89 + 704) + 325.76(98 + 372) + 68.44(704 + 19237) \approx \$3,171,900
\]  

We want to consider how the participation rates in transition programs affect this cost, so we write our function \( C \) as:

\[
C = f(q_1, q_2)
\]  

The concept of cost-effectiveness is used to compare strategies in terms of cost per inmate education achieved in implementing a particular strategy. In this case an inmate has the opportunity to get education classes while in prison and go through a transition program. The aim is to find which strategy is the most cost-effective when the Department of Corrections of Arizona is willing to spend a certain amount per unit increase in effectiveness. In general, finding the most cost-effective model is a two-step process [8]. First, we need to check if offering inside education only is most cost effective compared to offering inside education and the outside program and second, we try to find the most cost-effective strategy among all cost-effective strategies. Although the inside education program only strategy may prove more cost effective, it may not be as effective as our second option over a certain period of time since data suggest higher levels of recidivism for people who have only taken only education programs inside [1]. The inmates who take the education programs inside and when released go through the transition program seem to have a large cost. However, further studies have suggested that it is efficient in lowering recidivism [1].

Only counting the inside program we find from 2017 data:

\[
C = c_1(E^*_1 + E^*_2) + c_3(E^*_2 + I^*_2) = 2085.06(89 + 704) + 68.44(704 + 19237) \approx \$3,018,210
\]  

5 Results

In this section we provide the discussion, due to the selection process used we attempted to achieve a cost analysis for the implementation of Education programs with 2017 data. In addition we also obtained a numerical value for the reproductive number and we show numerical simulations of the model.

5.1 Numerical Analysis of \( R_e \)

Using the parameter calculations and the values for the variables found in the appendix we assigned numerical values to \( I_1, E_1, \beta_1, \beta_2, \gamma_2, \mu_2, \mu_4, \alpha_2 \). This allowed us to create a function for \( R_e \) in order to analyze the effects of our parameters \( p_2 \), the recidivism rate of educated prisoners, and \( q_2 \), the proportion of educated inmates that enter the GED program. The function for \( R_e \) is as follows

\[
R_e = \frac{0.310375}{1 - \frac{p_2(1 - 0.827586 + q_2)}{0.129 + p_2}}
\]  

We then graphed a contour plot and found the effect of recidivism on education. We found that as the recidivism rate increases \( R_e \) becomes greater than 1. The proportion of inmates that have a GED and leave the prison has to be below 0.35 otherwise \( R_e \) falls below 1. In our model, we want \( R_e \) to be greater than 1 because our infection is education and the goal is for people to get educated.
5.2 Equilibrium Analysis

We now look at the effect of education on recidivism. In order to observe this, we look at the two compartments that compose the recidivist class.

\[ R = E_1 + E_2 \]

In order to analyze the recidivist class we used the equilibrium points:

\[ E_1^* = \frac{r_1(\alpha_2(-i_2)+\lambda_1+\lambda_2)}{\beta_1(\alpha_2(-i_2)+\lambda_1+\lambda_2) + \alpha_1(1-r_1)} \]

\[ E_2^* = \frac{\alpha_1(\alpha_2^2 - \lambda_2)}{\beta_1(\alpha_2(-i_2)+\lambda_1+\lambda_2)} - I_2^* \]

\[ I_2^* \] is located in the Appendix A.2.

The following graph of this function gives us the behavior of the recidivism independent of the initial conditions of our system.

Figure 3: \( R_e \) Contour Plot

Figure 4: \( R_e \) Recidivism Behavior
We begin by looking at the behavior of $\gamma$ which represents the length of the transition program. Since we have an inverse proportion, we see that as the length of the transition program increases, the recidivist population is reduced.

For the proportion of uneducated inmates, we see that as the proportion of uneducated inmates going into the transition program gets larger, the recidivist population is also reduced.

Our graph also shows a point where the proportion of uneducated inmates leaving the prison and entering the transition program combined with a long enough program will make it possible to completely remove recidivism.

5.3 Cost Analysis

The cost function is $C = c_1(E_1^* + E_2^*) + c_2(T_1^* + T_2^*) + c_3(E_2^* + I_2^*)$. This represents the sum of the cost of recidivism ($c_1(E_1^* + E_2^*)$), cost of training program outside the jail ($c_2(T_1^* + T_2^*)$), and the cost of GED inside the prison ($c_3(E_2^* + I_2^*)$). This function depends on the number of released inmates ($q_1$ and $q_2$) going through the transition program.

If $q_1 = q_2 = 1$ (all released prisoners go through the transition programs), the total cost is:

$C = 2085.06(66 + 416) + 68.44(264 + 1745) + 68.44(416 + 19211) = 3.00272 \times 10^6$ Comparing to the original cost when $q_1$ and $q_2$ are different, there is $169,180 saving.

If $q_1 = q_2 = 0$ (Absence of transition programs so $T_1 = T_2 = 0$), the total cost is:

$C = 2085.06(104 + 808) + 68.44(808 + 19246) = 3.27407 \times 10^6$ We can notice that the cost is even more than the original, there is an increment of $102,170.

Since the cost depends on the quantities $q_1$ and $q_2$, and our ultimate goal is to minimize the cost, we graphed the cost function while varying both quantities ($q_1$ and $q_2$). The following graph depicts this variation and shows where the cost is at its minimum.

![Figure 5: Total cost as function of the participation on re-entry programs](image)

In figure we can see that that cost function is at its minimum when $q_2 = q_1 = 1$. This can be translated to universal transition program where all released inmates go through it. According to our model, this will lead to a minimum cost comparing to other situations. In addition to that, this has the potential to generate more income since people with higher education earn more on average which will result to higher contribution to taxes and less dependency on government assistance.

6 Conclusion

Education is typically defined as something that occurs during youth it does not become a choice until you become an adult. Education doesn’t need to be traditional it can also take on many different forms that involve learning. In our study we look at both aspects of education, the academic GED side which we suppose will give the inmate an advantage when it comes to job hunting, and the non-academic side or the transition program which creates opportunities for the parolee that will help the inmate avoid going to back to prison. Because of this, we have to look at $R_e$, the transmission of education, from a unique perspective. We assume that education would help a paroled inmate succeed and avoid returning to prison and with enough peer influence they will be convinced to join the educated inmates.

We calculated $R_e$ as being 1.175, which means that every educated inmate can influence at least 1
uneducated inmate. From the equilibrium points we reached the conclusion that it is possible to reduce recidivism to 0 given that enough uneducated inmates enter the transition program and the length of the program is long enough. Therefore we conclude that allocating more resources to the transition program will aid in helping the state of Arizona reduce recidivism.

In order to better make a statement about the cost we conducted a cost-analysis and found that transition programs in Arizona eventually pay for themselves because the total cost of is minimized when every released inmate enters a reentry program. This conclusion agrees with our conclusion above that reducing recidivism requires a large proportion of prisoners to enter the transition programs upon release. This proves our hypothesis that was postulated. Therefore we make the recommendation that more resources and funding needs to be allocated to transition programs. We also recommend that the state of Arizona conduct further studies in order to identify characteristics of successful transition programs in order to further strengthen the qualitative analysis of the programs.

There are other factors that affect recidivism like mandatory minimums, work-program agreements, and felony laws concerning sentencing. There are also demographic factors, length of time served, and type of offense that can be further explored within the population. Therefore, in order to strengthen our study we would want to incorporate more variables and parameters that can affect recidivism, not just education, and create a more exhaustive and comprehensive model that can truly help us understand the more complex dynamics of prisons.

7 Acknowledgments

We would like to thank Dr. Carlos Castillo-Chavez, Founding and Co-Director of the Mathematical and Theoretical Biology Institute (MTBI), for giving us the opportunity to participate in this research program. We would also like to thank Co-Director Dr. Anuj Mubayi as well as Coordinator Ms. Rebecca Perlin and Management Intern Ms. Sabrina Avila for their efforts in planning and executing the day to day activities of MTBI. We also want to give special thanks to Dr. Christopher M Kribs, Bechir Amdoundi, Dr. Naala Brewer, and Dr. Baltazar Espinoza. This research was conducted as part of 2018 MTBI at the Simon A. Levin Mathematical, Computational and Modeling Sciences Center (MCMSC) at Arizona State University (ASU). This project has been partially supported by grants from the National Science Foundation (NSF Grant MPS-DMS-1263374 and NSF Grant DMS-1757968), the National Security Agency (NSA Grant H98230-J8-1-0005), the Alfred P. Sloan Foundation, the Office of the President of ASU, and the Office of the Provost of ASU.
A Appendix

A.1 Appendix A: Education free equilibrium and $R_e$

This equilibrium only exits if we are assume $\Lambda_2 = 0$. Then, considering all the 8 equilibrium conditions, we got as a result that $E_2, O_2, T_2, I_2 = 0$ in order to stop the transmission: So, the remaining equations are:

1. $\frac{dI_1}{dt} = \Lambda_1 - \alpha_1 I_1$
2. $\frac{dE_1}{dt} = p_2 O_1 - \alpha_1 E_1$
3. $\frac{dT_1}{dt} = q_1 \alpha_1 (I_1 + E_1) - T_1 (\mu_1 + \gamma_1)$
4. $\frac{dO_1}{dt} = \gamma_1 T_1 + (1 - q_1) \alpha_1 (E_1 + I_1) - O_1 (\mu_3 + p_1)$

Using Mathematica we set the equations to zero and solve for the Education free equilibrium (EFE). After this, we get the following values and we added at star as an exponent to differentiate them.

\[
I_1^* = \frac{\Lambda_1}{\alpha_1}, \quad E_1^* = \frac{\Lambda_1 p_1}{\alpha_1 \mu_3 (\gamma_1 + \mu_1) + \mu_1 q_1 p_1}, \quad T_1^* = \frac{\Lambda_1 q_1 (p_1 + \mu_3)}{\mu_3 (\gamma_1 + \mu_1) + p_1 q_1 \mu_1}, \quad O_1^* = \frac{\Lambda_1 \mu_1 (1 - q_1 \mu_1) + \gamma_1}{\mu_3 (\gamma_1 + \mu_1) + p_1 q_1 \mu_1}
\]

To compute the basic reproductive number, we use the next generation operator. First, from the original equations of our model we create the $F$ vector that is based on new criminals infections that will help to find the $F$ matrix

\[
F = \begin{pmatrix}
\frac{\partial(x_1)}{\partial(I_2)} & \frac{\partial(x_1)}{\partial(E_2)} & \frac{\partial(x_1)}{\partial(T_2)} & \frac{\partial(x_1)}{\partial(O_2)} \\
\frac{\partial(x_2)}{\partial(I_2)} & \frac{\partial(x_2)}{\partial(E_2)} & \frac{\partial(x_2)}{\partial(T_2)} & \frac{\partial(x_2)}{\partial(O_2)} \\
\frac{\partial(x_3)}{\partial(I_2)} & \frac{\partial(x_3)}{\partial(E_2)} & \frac{\partial(x_3)}{\partial(T_2)} & \frac{\partial(x_3)}{\partial(O_2)} \\
\frac{\partial(x_4)}{\partial(I_2)} & \frac{\partial(x_4)}{\partial(E_2)} & \frac{\partial(x_4)}{\partial(T_2)} & \frac{\partial(x_4)}{\partial(O_2)}
\end{pmatrix} = \begin{pmatrix}
I_1 \beta_1 (I_2 + E_2) + \Lambda_2 \\
E_1 \beta_2 (I_2 + E_2) \\
0 \\
0
\end{pmatrix}
\]

Then, we compute the $F$ matrix

\[
F = \begin{pmatrix}
\frac{\partial(x_1)}{\partial(x_1)} & \frac{\partial(x_1)}{\partial(x_1)} & \frac{\partial(x_1)}{\partial(x_1)} & \frac{\partial(x_1)}{\partial(x_1)} \\
\frac{\partial(x_2)}{\partial(x_1)} & \frac{\partial(x_2)}{\partial(x_1)} & \frac{\partial(x_2)}{\partial(x_1)} & \frac{\partial(x_2)}{\partial(x_1)} \\
\frac{\partial(x_3)}{\partial(x_1)} & \frac{\partial(x_3)}{\partial(x_1)} & \frac{\partial(x_3)}{\partial(x_1)} & \frac{\partial(x_3)}{\partial(x_1)} \\
\frac{\partial(x_4)}{\partial(x_1)} & \frac{\partial(x_4)}{\partial(x_1)} & \frac{\partial(x_4)}{\partial(x_1)} & \frac{\partial(x_4)}{\partial(x_1)}
\end{pmatrix} = \begin{pmatrix}
\beta_1 i_1 & \beta_1 i_1 & 0 & 0 \\
\beta_2 e_1 & \beta_2 e_1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

From the terms of the original equations that are left, we are able to create the $V$ vector, which follows the inflow and outflow of criminals from each compartment
Here is the computation of the \( V \) matrix

\[
V = \begin{pmatrix}
\frac{\partial(y_1)}{\partial(y_1)} & \frac{\partial(y_1)}{\partial(y_2)} & \frac{\partial(y_1)}{\partial(y_3)} & \frac{\partial(y_1)}{\partial(y_4)} \\
\frac{\partial(y_2)}{\partial(y_1)} & \frac{\partial(y_2)}{\partial(y_2)} & \frac{\partial(y_2)}{\partial(y_3)} & \frac{\partial(y_2)}{\partial(y_4)} \\
\frac{\partial(y_3)}{\partial(y_1)} & \frac{\partial(y_3)}{\partial(y_2)} & \frac{\partial(y_3)}{\partial(y_3)} & \frac{\partial(y_3)}{\partial(y_4)} \\
\frac{\partial(y_4)}{\partial(y_1)} & \frac{\partial(y_4)}{\partial(y_2)} & \frac{\partial(y_4)}{\partial(y_3)} & \frac{\partial(y_4)}{\partial(y_4)}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\alpha_2 & 0 & 0 & 0 \\
0 & \alpha_2 & 0 & -p_2 \\
-\alpha_2 q_2 & -\alpha_2 q_2 & \gamma_2 + \mu_2 & 0 \\
\alpha_2 (q_2 - 1) & \alpha_2 (q_2 - 1) & -\gamma_2 & p_2 + \mu_4
\end{pmatrix}
\]

Then we compute the inverse of \( V \) using Wolfram Mathematica:

\[
V^{-1} = \frac{1}{\alpha_2}
\begin{pmatrix}
p_2 \gamma_2 & 0 & 0 & 0 \\
p_2 q_2 \gamma_2 & \alpha_2 (p_2 q_2 \mu_2 + (\gamma_2 + p_2) \mu_4) & 0 & 0 \\
\frac{\partial(y_1)}{\partial(y_1)} & \frac{\partial(y_1)}{\partial(y_2)} & \frac{\partial(y_1)}{\partial(y_3)} & \frac{\partial(y_1)}{\partial(y_4)} \\
\frac{\partial(y_2)}{\partial(y_1)} & \frac{\partial(y_2)}{\partial(y_2)} & \frac{\partial(y_2)}{\partial(y_3)} & \frac{\partial(y_2)}{\partial(y_4)} \\
\frac{\partial(y_3)}{\partial(y_1)} & \frac{\partial(y_3)}{\partial(y_2)} & \frac{\partial(y_3)}{\partial(y_3)} & \frac{\partial(y_3)}{\partial(y_4)} \\
\frac{\partial(y_4)}{\partial(y_1)} & \frac{\partial(y_4)}{\partial(y_2)} & \frac{\partial(y_4)}{\partial(y_3)} & \frac{\partial(y_4)}{\partial(y_4)}
\end{pmatrix}
\]

In order to find \( R_e \), we evaluate both matrices \( F \) and \( V^{-1} \) at our DFE and find the largest eigenvalue of the product of the two matrices

\[
R_e = \rho \left( F V^{-1} \right)
\]

\[
\rho = \begin{pmatrix}
\frac{I_1 \beta_1}{\alpha_2} + \frac{I_1 \beta_1 p_2 (\gamma_2 + \mu_2 - \mu_2 q_2)}{\alpha_2 (\mu_4 (\gamma_2 + \mu_2) \mu_4 + p_2 q_2)} & \frac{I_1 \beta_1 (\gamma_2 + \mu_2) (\mu_4 + p_2)}{\alpha_2 (\mu_4 (\gamma_2 + \mu_2) \mu_4 + p_2 q_2)} & \frac{I_1 \beta_1 \gamma_2 p_2}{\alpha_2 (\mu_4 (\gamma_2 + \mu_2) \mu_4 + p_2 q_2)} & \frac{I_1 \beta_1 p_2 (\gamma_2 + \mu_2)}{\alpha_2 (\mu_4 (\gamma_2 + \mu_2) \mu_4 + p_2 q_2)} \\
\frac{E_3 \beta_3}{\alpha_2} + \frac{E_3 \beta_3 p_2 (\gamma_2 + \mu_2 - \mu_2 q_2)}{\alpha_2 (\mu_4 (\gamma_2 + \mu_2) \mu_4 + p_2 q_2)} & \frac{E_3 \beta_3 (\gamma_2 + \mu_2) (\mu_4 + p_2)}{\alpha_2 (\mu_4 (\gamma_2 + \mu_2) \mu_4 + p_2 q_2)} & \frac{E_3 \beta_3 \gamma_2 p_2}{\alpha_2 (\mu_4 (\gamma_2 + \mu_2) \mu_4 + p_2 q_2)} & \frac{E_3 \beta_3 p_2 (\gamma_2 + \mu_2)}{\alpha_2 (\mu_4 (\gamma_2 + \mu_2) \mu_4 + p_2 q_2)} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
R_e = \frac{(\gamma_2 + \mu_2) (\mu_4 + p_2) (\beta_2 E_1 + \beta_1 I_1)}{\alpha_2 (\mu_4 (\gamma_2 + \mu_2) \mu_4 + p_2 q_2) q_2}
\]

This expression can be simplified by dividing whole expression of \( R_e \) by the constant part in the numerator

\[
\frac{\beta_2 E_1 + \beta_1 I_1}{\alpha_2 (\mu_4 (\gamma_2 + \mu_2) \mu_4 + p_2 q_2)}
\]

Manipulating the denominator(Den) with some algebra, we obtain a reduced expression of it. We can also apply the reciprocal of some terms and sum them all to get a final expression

\[
\text{Den} = \alpha_2 \left( \frac{\mu_2 p_2 q_2}{(\gamma_2 + \mu_2)(\mu_4 + p_2)} + 1 - \frac{p_2}{\mu_2 + \mu_4} \right)
\]

\[
\text{Den} = \alpha_2 \left( \frac{p_2 q_2}{(\gamma_2 + \mu_2)} \left( 1 - \frac{\gamma_2}{(\gamma_2 + \mu_2)} \right) + 1 - \frac{p_2}{\mu_2 + \mu_4} \right)
\]

\[
\text{Den} = \alpha_2 \left( \frac{p_2 q_2}{(\gamma_2 + \mu_2)} \left( 1 - \frac{\gamma_2}{(\gamma_2 + \mu_2)} \right) \right)
\]

After this, we can put the whole term together to lead up to:
\[ R_e = \left( \frac{I_1 \beta_1 + E_1 \beta_2}{\alpha_2} \right) \frac{1}{1 - \frac{p_1}{\mu_2 + \gamma_1} + \frac{p_2}{\mu_2 + \gamma_2}} \]

### A.2 Endemic Equilibrium

In this case, our recidivism model is considered with \( \Lambda > 0 \). So we arranged the original equations, the 8 equations we have below, in terms of \( I_2, I_1, E_2, E_1 \):

1. \[ \frac{dI_1}{dt} = \Lambda_1 - \beta_1 I_1 (I_2 + E_2) - \alpha_1 I_1 \]
2. \[ \frac{dI_2}{dt} = \Lambda_2 + \beta_1 I_1 (I_2 + E_2) - \alpha_2 I_2 \]
3. \[ \frac{dE_1}{dt} = p_1 O_1 - \beta_2 E_1 (I_2 + E_2) - \alpha_1 E_1 \]
4. \[ \frac{dE_2}{dt} = p_2 O_2 + \beta_2 E_1 (I_2 + E_2) - \alpha_2 E_2 \]
5. \[ \frac{dT_1}{dt} = (I_1 + E_1)(q_1 \alpha_1) - T_1 (\mu_1 + \gamma_1) \]
6. \[ \frac{dT_2}{dt} = (I_2 + E_2)(q_2 \alpha_2) - T_2 (\mu_2 + \gamma_2) \]
7. \[ \frac{dO_1}{dt} = \gamma_1 T_1 + (1 - q_1)(\alpha_1 I_1 + \alpha_1 E_1) - O_1 (p_1 + \mu_3) \]
8. \[ \frac{dO_2}{dt} = \gamma_2 T_2 + (1 - q_2)(\alpha_2 I_2 + \alpha_2 E_2) - O_2 (p_2 + \mu_4) \]

We added \( e \) as an exponent to denote the new equations we got for the endemic equilibrium

\[
T_1^e = \frac{q_1 \alpha_1}{\mu_1 + \gamma_1} (I_1^e + E_1^e) \\
T_2^e = \frac{q_2 \alpha_2}{\mu_2 + \gamma_2} (I_2^e + E_2^e) \\
O_1^e = \frac{\alpha_1}{\mu_3 + p_1} [(1 - q_1) + q_2 \frac{\gamma_1}{\mu_1 + \gamma_1}] (I_1^e + E_1^e) \\
O_2^e = \frac{\alpha_2}{\mu_4 + p_2} [(1 - q_2) + q_2 \frac{\gamma_2}{\mu_2 + \gamma_2}] (I_2^e + E_2^e) \\
\Lambda_1 = I_1^e (\alpha_1 + \beta_1 (I_2^e + E_2^e)) \\
\Lambda_2 = \alpha_2 I_2^e - \beta_1 I_1^e (I_2^e + E_2^e) \\
p_1 O_1^e = E_1^e (\alpha_1 + \beta_1 (I_2^e + E_2^e)) \\
\alpha_2 E_2^e = p_2 O_2^e + \beta_2 E_1^e (I_2^e + E_2^e) \]

Next, we can add the 2 equations for \( O_1 \):

\[
\alpha_1 \frac{p_1}{\mu_3 + p_1} [1 - \frac{q_1 \mu_1}{\mu_1 + \gamma_1}] (I_1^e + E_1^e) = E_1^e (\alpha_1 + \beta_1 (I_2^e + E_2^e)) \tag{9}
\]

and calling \( r_1 = \frac{p_1}{\mu_3 + p_1} [1 - \frac{q_1 \mu_1}{\mu_1 + \gamma_1}] \), to make it simple for later.

Same process for the 2 more for \( O_2 \):

\[
\alpha_2 \frac{p_2}{\mu_4 + p_2} [1 - \frac{q_2 \mu_2}{\mu_2 + \gamma_2}] (I_2^e + E_2^e) + \beta_2 E_1^e (I_2^e + E_2^e) = \alpha_2 E_2^e \tag{10}
\]

with \( r_2 = \frac{p_2}{\mu_4 + p_2} [1 - \frac{q_2 \mu_2}{\mu_2 + \gamma_2}] \)

For the next step we simplify and add \( O_1 \) and \( O_2 \) and setting the respective \( r_1 \) and \( r_2 \) we get:

\[
\alpha_1 r_1 I_1^e + \alpha_2 r_2 I_2^e = \alpha_1 (1 - r_1) E_1^e + \alpha_2 (1 - r_2) E_2^e
\]
In order to derive $E_1^c$:

$$
\alpha_1 r_1 I_1^c = \alpha_1 (1 - r_1) E_1^c + \beta_2 E_1^c (I_2^c + E_2^c)
$$

$$
E_1^c = \frac{r_1 (\Lambda_1 + \Lambda_2 - \alpha_2 I_2^c)}{\alpha_1 (1 - r_1) + \beta_2 (I_2^c + E_2^c)}
$$

We also added $\Lambda_1$ and $\Lambda_2$ in order to get a value for $I_1^c$

$$
I_1^c = \frac{\Lambda_1 + \Lambda_2 - \alpha_2 I_2^c}{\alpha_1}
$$

This will help us to find $E_2^c$,

$$
E_2^c = \frac{\alpha_1(\alpha_2 I_2^c - \Lambda_2)}{\beta_1(\Lambda_1 + \Lambda_2 - \alpha_2 I_2^c)} - I_2^c
$$

Now, we can plug in $E_2^c$ into $I_1^c$ and add equations 9 and 10 to simplify some terms and end up with this:

$$
r_1(\Lambda_1 + \Lambda_2) + (1 - r_1)\alpha_2 I_2^c = \frac{r_1(1 - r_1)(\Lambda_1 + \Lambda_2 - \alpha_2 I_2^c)^2}{(1 - r_1)(\Lambda_1 + \Lambda_2 - \alpha_2 I_2^c) + \beta_2/\beta_1(\alpha_2 I_2^c - \Lambda_2)} + \frac{\alpha_1\alpha_2(1 - r_2)(\alpha_2 I_2^c - \Lambda_2)}{\beta_1(\Lambda_1 + \Lambda_2 - \alpha_2 I_2^c)}
$$

setting $x = -\Lambda_2 + \alpha_2 I_2^c$ we obtain a cubic equation that can be solve in Wolfram Mathematica numerically, just plugging in the parameters values mentioned in previous sections.

$$
(\Lambda_2 + \Lambda_1 r_1 + (1 - r_1)x)((1 - r_1)(\Lambda_1 - x) + \frac{\beta_2x}{\beta_1})(\Lambda_1 - x) = r_1(1 - r_1)(\Lambda_1 - x)^3 + \frac{\alpha_1\alpha_2}{\beta_1}(1 - r_2)x((1 - r_1)(\Lambda_1 - x) + (\beta_2/\beta_1)x)
$$

Using Mathematica to solve this cubic equation will lead you to 3 values for $I_2^c$ but the one that works for this study is just when $I_2^c = 19236.9$ because, it’s the value that give us all positive numbers whenever we replace it on the other equations expressed in terms of $I_2$. The following values for the other variables of the endemic equilibrium after replacing the parameters values are:

$$
I_1^c = \frac{-\alpha_2 I_2^c + \lambda_1 + \lambda_2}{\alpha_1} = 2088.1
$$

$$
E_2^c = \frac{\alpha_1(\alpha_2 I_2^c - \lambda_2)}{\beta_1(-\alpha_2 I_2^c + \lambda_1 + \lambda_2)} - I_2 = 89
$$

$$
E_1^c = \frac{r_1(-\alpha_2 I_2^c + \lambda_1 + \lambda_2)}{\beta_2(E_2^c + I_2^c) + \alpha_1(1 - r_1)} = 704.4
$$

$$
T_1^c = \frac{(E_1^c + I_1^c)(\alpha_1 q_1)}{\gamma_1 + \mu_1} = 98
$$

$$
T_2^c = \frac{(E_2^c + I_2^c)(\alpha_2 q_2)}{\gamma_2 + \mu_2} = 372
$$

$$
O_1^c = \frac{\alpha_1 E_1^c - \beta_2 E_1^c (E_2^c + I_2^c)}{p_1} = 1,050
$$

$$
O_2^c = \frac{\alpha_2 e_2 - \beta_2 e_1 (e_2 + i_2)}{p_2} = 2,149
$$

These values are the ones which allow us to construct the cost analysis.
References


https://students.asu.edu/standard-cost-attendance
